Gauge invariance and classical dynamics of noncommtative particle theory

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Abstract

We consider a model of classical noncommutative particle in an external electromagnetic field. For this model, we prove the existence of generalized gauge transformations. Classical dynamics in Hamiltonian and Lagrangian form is discussed, in particular, the motion in the constant magnetic field is studied in detail.

1 Introduction

In the last decade, there has been a certain interest in considering quantum-mechanical and field-theoretical models with noncommutative space-time coordinates, see e.g. [1] and [2] for reviews on noncommutativity in QFT and QM, respectively. The noncommutative space can be realized by the coordinates $\hat{x}^i$, satisfying commutation relations $[\hat{x}^i, \hat{x}^j] = i\theta^{ij}$, where θ^{ij} is an antisymmetric constant matrix. Classical actions of field theories on a noncommutative space-time can be written as some modified classical actions already on the commutative space-time, using the Weyl-Moyal correspondence [3]. Similar possibility exists in the case of a finite-dimensional theory (mechanics). The noncommutativity of position coordinates can be obtained as a consequence of a canonical quantization of dynamical models [4]- [10]. For example, the canonical quantization of the classical theory with first-order action

$$S_{NC} = S_H + S_\theta, \tag{1}$$

$$S_H = \int dt \left[p_j \dot{x}^j - H(p, x) \right], \quad S_\theta = \int dt p_i \theta^{ij} \dot{p}_j / 2,$$

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leads to a quantum theory (which is called noncommutative quantum mechanics (NCQM)) with the commutation relations

$$\left[\hat{x}^i, \hat{x}^j\right] = i\theta^{ij}, \quad \left[\hat{x}^i, \hat{p}_j\right] = i\delta^i_j, \quad [\hat{p}_i, \hat{p}_j] = 0, \tag{2}$$

and with the quantum Hamiltonian $H(\hat{x}, \hat{p})$. The model of noncommutative particle (1) was proposed in [5], see also [6]. In fact, S_H is the ordinary Hamiltonian action and S_θ is responsible for noncommutativity.

As it was already mentioned, NCQM has been studied extensively [2], and many calculations on the base of the theory were performed to find the upper bound on the noncommutativity parameter θ. However, there remain some open questions in classical mechanics of noncommutative particle, for example, the problem of gauge invarience with respect to the external electromagnetic field. It is known that NCQM with external electromagnetic field is invariant under the noncommutative $U(1)$ gauge group, which is $U_\star(1)$, see [11]. This fact may serve as an indication that there exist a classical version of such transformations. In fact, the problem is closely related to the problem of introducing the interaction with the Abelian gauge field in the classical models of noncommutative particle (1), see e.g. [12].

As it is now known, there exist two ways of introducing potentials $A_\mu(x) = (A_i(x), A_0(x) = \varphi(x), \; i = 1, ..., n)$ of the external electromagnetic field in the theory, which correspond to two different actions $S^1_{NC} = S^1_H + S_\theta$ of the Duval-Horvathy model [5], and $S^2_{NC} = S^2_H + S_\theta$ of the Deriglazov model [6], where

$$S^1_H = \int dt \left[(p_j + eA_j(x)) \dot{x}^j - \frac{1}{2}p^2 - e\varphi(x) \right], \tag{3}$$

$$S^2_H = \int dt \left[p_j \dot{x}^j - \frac{1}{2} [p_i - eA_i(x)]^2 - e\varphi(x) \right] \tag{4}$$

The action S^1_{NC}, by the construction, is invariant under the $U(1)$ gauge transformations: $\delta A_i = \partial_i f(x)$ and the particle momenta have not to be changed. Classical equations of motion describing dynamics of noncommutative particle in Duval-Horvathy model were investigated in details in [13]. Hamiltonization of the theory with the action S^1_{NC} leads to the following Dirac brackets between the phase space variables x^i, p_j:

$$\left\{x^i, x^j\right\}_D = \varepsilon^{ij}\theta d, \quad \left\{x^i, p_j\right\}_{D(\Phi)} = \delta^i_j d,$$
$$\{p_i, p_j\}_D = \varepsilon_{ij}eBd, \quad d = 1 - e\theta B(x), \tag{5}$$

where $B(x) = \partial_1 A_2 - \partial_2 A_1$ is the magnetic field. After quantization they determine the commutation relations between coordinates and momenta:

$$\left[\hat{x}^i, \hat{x}^j\right] = i\theta\varepsilon^{ij}d, \quad \left[\hat{x}^i, \hat{p}_j\right] = i\delta^i_j d, \quad [\hat{p}_i, \hat{p}_j] = ieB\varepsilon_{ij}d. \tag{6}$$

As it was first mentioned in [8] the space noncommutativity depends on magnetic field $B(x)$ and this is not a case of usually considered noncommutativity with constant θ. The canonical quantization of the Deriglazov model leads to the space noncommutativity with a constant θ. That is why we concentrate our attention on the Deriglazov model in what follows.

In spite of the fact that the action (4) is invariant under the standard gauge transformations:

$$\delta A_i = \partial_i f(x), \delta p_i = e\partial_i f(x), \tag{7}$$

the complete action S^2_{NC} is not, due to the term S_θ. In the work [12], on the example of planar particle, $n = 2$, only infinitesimal local transformations

$$\delta x^i = -e\theta\varepsilon^{ij}\partial_j\Lambda(x), \quad \delta p_i = e\partial_i\Lambda(x),$$
$$\delta A_i = A_i'(x + \delta x) - A_i(x) = \partial_i\Lambda(x), \tag{8}$$

were constructed, which preserve simplectic structure of S^2_{NC}, and change corresponding Lagrangian on the total time derivative.

In the present article we demonstrate the existance of generalized gauge transformations for the Deriglazov model. These transformations are deformation in θ of the gauge transformations (7). In the first order in θ they coincide with (8). After quantization the generalized gauge transformations lead to the gauge group of NCQM, see [11]. Then, we consider classical dynamics of the model in the configuration space, and a possibility to construct a Lagrangian second-order action which is equivalent to the Hamiltonian first-order action (1). The general consideration is illustrated by an example of the noncommutative charged particle in a constant magnetic field.

2 Generalized gauge transformations

The action S^2_{NC} can be written as follows:

$$S^2_{NC} = \int dt L^\theta_H, \quad L^\theta_H = L_1 - H,$$
$$L_1 = p_j \dot{x}^j + \frac{1}{2}p_i\theta^{ij}\dot{p}_j, \quad H = \frac{1}{2}(p_i - eA_i(x))^2 + e\varphi(x). \tag{9}$$

The simplectic structure (Poisson brackets) corresponding to this first-order action is:

$$\{x^i, x^j\} = \theta^{ij}, \quad \{x^i, p_j\} = \delta^i_j, \quad \{p_i, p_j\} = 0. \tag{10}$$

Below, we are going to construct an explicit form of the generalized gauge transformations. To this end, first, we introduce the following transformations:

$$\delta x^i = K^i(x), \quad \delta p_i = J_i(x), \quad J_i(x) = e\partial_i f(x) + O(\theta), \tag{11}$$

which should leave the simplectic structure (10) invariant. That is, new coordinates and momenta

$$x'^i = x^i + K^i(x), \quad p'_i = p_i + J_i(x) \tag{12}$$

must have the same Poisson brackets. From this condition one obtains equations on the functions $K^i(x)$ and $J^i(x)$:

$$\theta^{il}\partial_l K^j - \theta^{jl}\partial_l K^i + \{K^i, K^j\} = 0,$$
$$\theta^{il}\partial_l J_j - \partial_j K^i + \{K^i, J_j\} = 0, \quad \partial_i J_j - \partial_j J_i + \{J_i, J_j\} = 0, \tag{13}$$

where the Poisson brackets between two functions of coordinates are determined as

$$\{F, G\} = (\partial_k F)\,\theta^{kl}\,(\partial_l G).$$

If $K^i = -\theta^{il}J_l$ then two first equations (13) are just the consequences of the third one.

Thus, to find (11) we have to solve the following differential equation:

$$\partial_j J_i - \partial_i J_j = \{J_i, J_j\}, \tag{14}$$

with the condition that $J_i(x) = e\partial_i f(x) + O(\theta)$. The solution of this equation can be found as a perturbative series in θ, and has the form

$$J_i(x) = \sum_{m=0}^{\infty} \frac{e^{m+1}}{(m+1)!}\underbrace{\{...\{\partial_i f, \underbrace{f\}, ..., f\}}_{m}}_{m} = \sum_{m=0}^{\infty} J_i^m(x), \tag{15}$$

where[1]

$$J_i^m(x) = \frac{e}{m+1}\left\{J_i^{m-1}, f\right\}, \quad m \geq 1, \tag{16}$$
$$J_i^0(x) = e\partial_i f(x).$$

Let us prove it by the induction. One can easily verify that

$$J_i^1(x) = \frac{e^2}{2}\{\partial_i f, f\} = \frac{e}{2}\left\{J_i^0, f\right\}$$

is the solution of the equation (14) in the first order in θ. We should prove that if $J_i^m(x)$ is the solution of this equation in the m-th order, i.e.,

$$\partial_j J_i^m - \partial_i J_j^m = \sum_{l=0}^{m-1}\left\{J_i^{m-1-l}, J_j^l\right\}, \tag{17}$$

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[1] By the construction, the function $J_i^m(x)$ is of the m-th order in θ.

holds, then the solution in the order $m + 1$ is:

$$J_i^{m+1}(x) = \frac{e}{m+2} \{J_i^m, f\}. \tag{18}$$

Let us consider the following quantity

$$I_{ij} = (m+2)\left(\partial_j J_i^{m+1} - \partial_i J_j^{m+1}\right) = e\left(\partial_j \{J_i^m, f\} - \partial_i \{J_j^m, f\}\right).$$

With the help of (17), it can be rewritten as

$$I_{ij} = \sum_{l=0}^{m-1} e\left\{\left\{J_i^{m-1-l}, J_j^l\right\}, f\right\} + \{J_i^m, e\partial_j f\} + \{e\partial_i f, J_j^m\}.$$

Using the Jacobi identity and (16), we reduce I_{ij} to the following form

$$\sum_{l=0}^{m-1}\left[(m-l+1)\left\{J_i^{m-l}, J_j^l\right\} + (l+2)\left\{J_i^{m-1-l}, J_j^{l+1}\right\}\right] + \left\{J_i^m, J_j^0\right\} + \left\{J_i^0, J_j^m\right\}$$

$$= \sum_{l=0}^{m-1}\left[(m-l+1)\left\{J_i^{m-l}, J_j^l\right\} + (l+1)\left\{J_i^{m-l}, J_j^l\right\}\right] + (m+2)\left\{J_i^0, J_j^m\right\}$$

$$= (m+2)\sum_{l=0}^{m}\left\{J_i^{m-l}, J_j^l\right\},$$

and prove therefore that

$$\partial_j J_i^{m+1} - \partial_i J_j^{m+1} = \sum_{m=0}^{m}\left\{J_i^{m-1-m}, J_j^m\right\}.$$

In turn, this means that (18) is a solution of equation (14) in $(m+1)$-th order with respect to θ.

Finally we obtain:

$$\delta x^i = K^i(x) = -\sum_{m=1}^{\infty}\frac{e^m}{m!}\underbrace{\{...\{x^i, \underbrace{f\}, ..., f\}}_{m}}_{m},$$

$$\delta p_i = J_i(x) = \sum_{m=0}^{\infty}\frac{e^{m+1}}{(m+1)!}\underbrace{\{...\{\partial_i f, \underbrace{f\}, ..., f\}}_{m}}_{m}. \tag{19}$$

The invariance of the Hamiltonian H from (9) under the transformations (19) implies the generalized gauge transformation of the potential $A_\mu(x)$:

$$A_i \rightarrow A_i'(x^i + \delta x^i) = A_i(x) + \frac{1}{e}\delta p_i,$$

$$\varphi \rightarrow \varphi'\left(x + \delta x^i\right) = \varphi(x). \tag{20}$$

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An explicit form of the transformed potential $A_i'(x)$ can be obtained by iterating the relation

$$A_i'(x) = A_i(x) + \frac{1}{e}\delta p_i - \sum_{m=1}^{\infty} \frac{e^m}{m!} \partial_{i_1}...\partial_{i_m} A_i'(x) \, \delta x^{i_1}...\delta x^{i_m}.$$

Up to the first order, one can obtain

$$A_i'(x) = A_i(x) + \partial_i f + e\{A_i + 3/2\partial_i f, f\}, \quad \varphi'(x) = \varphi(x) + \{\varphi, f\}.$$

That is,

$$\delta A_i = A_i'(x) - A_i(x) = \partial_i f + e\{A_i + 3/2\partial_i f, f\} + o(\theta),$$
$$\delta\varphi = \varphi'(x) - \varphi(x) = \{\varphi, f\} + o(\theta).$$

Since both Poisson brackets (10) and the Hamiltonian (9) are invariant under the transformations (19), (20), the corresponding classical dynamics

$$\dot{x}^i = \{x^i, H\}, \quad \dot{p}_i = \{p_i, H\}, \tag{21}$$

is invariant under these transformations.

Following [12], we introduce non-Abelian field strength

$$F_{ij}^\theta = \{p_i - eA_i(x), p_j - eA_j(x)\} = \partial_i A_j - \partial_j A_i + e\{A_i, A_j\}. \tag{22}$$

By the definition, it is invariant under the generalized gauge transformations (19) and (20). After the quantization the corresponding field strength is determined by

$$F_{ij}^\star = \partial_i A_j - \partial_j A_i + e[A_i, A_j]_\star, \tag{23}$$

where $F_{ij}^\star$ is the strength tensor of a gauge field related to non-Abelian $U_\star(1)$ group, the latter is the gauge group of NCQM, see [11].

As an example, we consider the case $n = 2$, $\varphi = 0$, and $A_i = (-By/2, Bx/2)$, which corresponds to a planar particle in a constant magnetic field. Let $f = Bxy/2$, then (15) reads,

$$J_1 = ay, \quad J_2 = bx,$$
$$a = \frac{eB}{2} - \frac{2 - \sqrt{e^2 B^2 \theta^2 + 4}}{2\theta}, \quad b = \frac{eB}{2} + \frac{2 - \sqrt{e^2 B^2 \theta^2 + 4}}{2\theta}.$$

Using formulas (19) and (20), we find the following gauge transformations

$$x \to x' = (1 - \theta b)x, \quad y \to y' = (1 + \theta a)y,$$
$$p_1 \to p_1' = p_1 + ay, \quad p_2 \to p_2' = p_2 + bx,$$
$$A_1 \to A_1' = \frac{2a - eB}{2e(1 + \theta a)}y', \quad A_2 \to A_2' = \frac{2b + eB}{2e(1 - \theta b)}x'. \tag{24}$$

One can easily verify that the corresponding variation of the Lagrangian (9) is reduced to a total derivative,

$$\delta L_H^\theta = \frac{d}{dt}\left[\frac{1}{2}\left(a - b\right)xy + \theta a p_2 y - \theta b p_1 x\right].$$

In the limit $\theta \to 0$, transformations (24) are reduced to the gradient gauge transformations $A_i \to A_i' = (0, Bx)$.

3 Dynamics in configuration space and Lagrangian action

Considering the Deriglazov model, we introduce new variables: $(x^i, p_i) \to (x^i, \pi_i)$, where $\pi_i = p_i - eA_i$. Poisson brackets involving new variables are

$$\left\{x^i, \pi_j\right\} = \delta_j^i - e\theta^{ik}\partial_k A_j, \quad \left\{\pi_i, \pi_j\right\} = eF_{ij}^\theta, \tag{25}$$

such that the equations of motion take the form

$$\begin{aligned}
\dot{x}^i &= \left\{x^i, H\right\} = \left(\delta_j^i - e\theta^{ik}\partial_k A_j\right)\pi_j + e\theta^{ij}\partial_j\varphi, \\
\dot{\pi}_i &= \left\{\pi_i, H\right\} = eF_{ij}^\theta\pi_j - e\left(\delta_j^i - e\theta^{ik}\partial_k A_j\right)\partial_j\varphi, \\
H &= \pi^2/2 + \varphi\left(x\right).
\end{aligned} \tag{26}$$

Excluding momenta π_i from equations (26), we obtain second-order equations of motion for the coordinates x^i. For simplicity, let us set $\varphi\left(x\right) = 0$, and $A_i\left(x\right)$ to be an arbitrary function of the coordinates. Then we obtain θ-modified Lorentz equations in the case under consideration,

$$\begin{aligned}
\ddot{x}^i &= F_\theta^i + \tilde{F}^i, \\
F_\theta^i &= eF_{ij}^\theta\dot{x}^j, \quad \tilde{F}^i = e\theta^{ki}\partial_k\partial_l A_j\left(\delta_m^j - e\theta^{jn}\partial_n A_m\right)^{-1}\dot{x}^l\dot{x}^m.
\end{aligned} \tag{27}$$

If $\theta = 0$, the equations are reduced to the ordinary Lorentz equations. If $\theta \neq 0$, the Lorentz force F_θ^i is changed, according to (22), and a new force $\tilde{F}^i$ proportional to square of velocities appears .

In the case of linear potential A_i, the term $\tilde{F}^i$ vanishes and F_{ij}^θ is just a constant. If $n = 2$ and $A_i = (-By/2, Bx/2)$ (the above considered magnetic field), equations (27) take the form:

$$\ddot{x} = e\tilde{B}\dot{y}, \quad \ddot{y} = -e\tilde{B}\dot{x}, \quad \tilde{B} = B\left(1 + e\theta B/4\right). \tag{28}$$

Its solutions were analyzed in [14].

Now we set $A_i = 0$ and $\varphi\left(x\right)$ to be an arbitrary function. In this case equations (26) yield

$$\ddot{x}^i - e\theta^{ij}\partial_j\partial_k\varphi\dot{x}^k + e\partial_i\varphi = 0. \tag{29}$$

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Considering $n = 2$ and $\varphi = \omega^2 \left(x^2 + y^2 \right) / 2$, we obtain:

$$\ddot{x} - e\theta\omega^2 \dot{y} + e\omega^2 x = 0 \ , \quad \ddot{y} + e\theta\omega^2 x + e\omega^2 y = 0 \ .$$

The latter equations coincide with equations of motion of a charge in a constant magnetic field $B_\theta = \theta\omega^2$ and linear electric field $\mathbf{E} = \left(\omega^2 x, \omega^2 y \right)$, i.e., in this case noncommutativity is equivalent to the presence of a magnetic field.

If $\varphi = y^2 / 2$, equations (29) read

$$\ddot{x} - e\theta\dot{y} = 0 \ , \quad \ddot{y} + ey = 0 \ .$$

This is a second-order non-Lagrangian set of equations, which does not admit an integrating multiplier, see [15, 16].

If $n = 2$, $A_i = \left(-By/2, Bx/2 \right)$ and $\varphi = \omega^2 \left(x^2 + y^2 \right) / 2$, we have

$$\ddot{x}^i - e \left(\tilde{B} + \theta\omega^2 \right) \varepsilon^{ij} \dot{x}^j + e\omega^2 \left(1 + \frac{e^2 \theta^2 B^2}{2} \right) x^i = 0 \ .$$

These equations coincide with equations of motion of a charge in a constant magnetic field $B_\theta = \tilde{B} + \theta\omega^2$ and a linear electric field $\mathbf{E} = \omega^2 \left(1 + e^2 \theta^2 B^2 / 2 \right) \left(x, y \right)$. For $\theta = -4B / \left(4\omega^2 + eB^2 \right)$, the effective magnetic field B_θ disappears.

In fact, the noncommutative particle action (1) is a first-order action, and can be treated as a Hamiltonian action. To construct a second-order Lagrangian formulation, we pass to Darboux coordinates. Namely, we change the variables as follows: $\left(x^i, p_i \right) \to \left(q^i, p_i \right)$, where

$$q^i = x^i + \frac{1}{2}\theta^{ij} p_j \ . \tag{30}$$

In the new variables, the action (1) takes the form

$$S^\theta \left[q, p \right] = \int dt \left[p_i \dot{q}^i - H \left(q^i - \theta^{ij} p_j / 2, p_i \right) \right] \ , \tag{31}$$

where $H \left(x, p \right)$ is defined in (9). From the equations

$$\frac{\delta S^\theta \left[q, p \right]}{\delta p_i} = 0 \Rightarrow \dot{q}^i = \frac{\partial H}{\partial p_i} \tag{32}$$

one can express the momenta p_i via coordinates q^i and velocities $\dot{q}^i$:

$$p_i = \dot{q}^i + eA_i \left(q \right) - e\partial_j A_i \left(q \right) \theta^{jk} \left[\dot{q}^k + eA_k \left(q \right) \right] \tag{33}$$
$$+ e\theta^{ij} \partial_j \varphi \left(q \right) + e\theta^{ij} \partial_j A_k \left(q \right) \dot{q}^k + o \left(\theta \right) .$$

Substituting (33) into (31), we obtain a second-order Lagrangian action S_L^θ that does not contain any momenta,

$$S_L^\theta = \int dt L^\theta, \quad L^\theta = \frac{1}{2}\dot{q}^2 + eA_i \dot{q}^i - e\varphi \left(q \right)$$
$$- e\dot{q}^i \partial_j A_i \theta^{jk} \left(\dot{q}^k + eA_k \right) - e^2 \partial_i \varphi \theta^{ij} A_j + o \left(\theta \right) .$$

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Such a form of noncommutative particle actions can be useful both for constructing Lagrangian path integrals in noncommutative quantum mechanics, and for searching integrals of motion. E.g., having the Lagrangian L^θ, we easily obtain the conserved energy

$$E_\theta = \frac{1}{2}\dot{q}^2 + e\varphi(q) + e^2\partial_i\varphi\theta^{ij}A_j - e\dot{q}^i\frac{\partial A_i}{\partial q^j}\theta^{jk}\dot{q}^k + o(\theta).$$

Let us consider the above construction for a specific case where $n = 2$, $A_i = (-By/2, Bx/2)$, and $\varphi = \omega^2(x^2 + y^2)/2$. In this case equations (32) can be solved exactly. Thus, we obtain

$$L^\theta = \varkappa\left[\left(\dot{q}_x^2 + \dot{q}_y^2\right) + e\left(\tilde{B} + \omega^2\theta/4\right)(q_x\dot{q}_y - q_y\dot{q}_x) - e\omega^2\left(q_x^2 + q_y^2\right)\right],$$

where

$$\varkappa = \left(2 + e^2B^2\theta^2/8 + eB\theta + e\omega^2\theta^2/2\right)^{-1}.$$

The corresponding θ-modification of the usual conserved energy $E_0 = \left(\dot{q}_x^2 + \dot{q}_y^2\right)/2 + e\omega^2\left(q_x^2 + q_y^2\right)/2$ is reduced to a multiplication by a factor, $E_\theta = 2\varkappa E_0$.

Acknowledgements

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